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A REVIEW OF BOUNDARY-LAYER LITERATURE

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SUMMARY

A concise nonmathematical review of the subject of boundary layers is presented. The contents, although insufficient for the solution of specific problems, are sufficient for an introduction to the subject. A list of reference papers is given from which the detailed knowledge necessary for the solution of specific problems can be obtained.

INTRODUCTION

The literature on the subject of boundary layers contains so many papers of varying quality that it is difficult for a newcomer to the subject to choose the papers that provide the maximum gain in knowledge for the effort expended. Although references 1 and 2 provide detailed reviews of boundary-layer theory, no short nonmathematical summary is readily available.

The purpose of the present paper is to provide a short summary that contains exact or approximate information that is believed to be useful. The summary is confined to cases for which the physical properties of the fluid are constant, that is, to incompressible flow with no temperature effects. An introduction to boundary-layer literature is provided, and reference papers are listed from which information on subjects of special interest may be obtained.

The material presented herein was originally presented as a talk at Wright Field, Dayton, Ohio, on September 26, 1945.

BOUNDARY-LAYER TERMINOLOGY

Symbols

ρ density

u velocity component parallel to surface

v velocity component perpendicular to surface

x distance along surface from leading edge

y distance normal to surface

p static pressure

μ coefficient of viscosity

U velocity component parallel to surface at outer edge of boundary layer

θ momentum thickness $\left(\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right)$

δ^* displacement thickness $\left(\int_0^\delta \left(1 - \frac{u}{U}\right) dy\right)$

δ nominal thickness of boundary layer

τ_o surface shearing stress

H ratio of displacement thickness to momentum thickness $\left(\frac{\delta^*}{\theta}\right)$

q dynamic pressure $\left(\frac{\rho U^2}{2}\right)$

$\phi = \left(\frac{\tau_o}{\rho U^2}\right)\left(\frac{Ux}{\upsilon}\right)^{1/2}$

υ kinematic viscosity $\left(\frac{\mu}{\rho}\right)$

U_o free-stream velocity

$\left|\frac{dU}{dx}\right|$ absolute magnitude of rate of change of U with x

R_x Reynolds number $\left(\frac{U_o x}{\upsilon}\right)$

c chord

R_c Reynolds number $\left(\frac{U_o c}{\upsilon}\right)$

τ local shearing stress

l mixing length

$v_* = \sqrt{\frac{\tau_o}{\rho}}$

m exponent in formula for boundary-layer velocity distribution

u_θ value of u at $y = \theta$

$\eta_1 = 1 - \left(\frac{u_\theta}{U}\right)^2$

R_δ Reynolds number $\left(\frac{U\delta}{\upsilon}\right)$

n exponent in formula for variation of velocity along surface

$X' = \frac{dX}{dx}$

$X = \frac{x}{U} \frac{dU}{dx}$

k constant in formula for variation of velocity along surface

$\overline{u'v'} = \lim_{t \to \infty} \left(\frac{1}{t} \int_0^t u'v' \, dt\right)$

u' x-component of fluctuation velocity

v' y-component of fluctuation velocity

t time

K von Kármán's universal constant

R_ϵ Reynolds number $\left(\frac{U\epsilon}{\nu}\right)$

ϵ height of roughness particle

ω angular velocity

a radius of disk

δ_* thickness of laminar sublayer

D constant in surface-friction formula

g constant in surface-friction formula

λ Pohlhausen shape parameter $\left(\frac{\delta^2}{\nu}\frac{dU}{dx}\right)$

C inflow velocity

Terminology

A boundary layer may be defined as a region in the flow field in which the viscous forces in the equation of motion are not all negligible. (See appendix.) For flow over bodies at the Reynolds numbers encountered in applied aerodynamics, the region in which the viscous forces are not negligible is confined to a thin layer of fluid next to the surface, the Prandtl boundary layer (reference 1). The viscous forces are confined to this thin boundary layer because the space rates of change of shearing stress are large enough to produce other than negligible viscous forces only in the thin surface layer of fluid. In the absence of solid boundaries, boundary layers occur where streams of fluid that move with different velocities are in contact; for example, jets and wakes. Boundary layers may be divided into two classes, laminar and turbulent.

A laminar boundary layer is one in which the paths of the particles of fluid never cross one another; the neighboring layers of fluid glide over one another as if they were solid sheets and all interchange of momentum between adjacent layers takes place only by molecular motions. A turbulent boundary layer, on the other hand, is one in which the paths of the particles of fluid cross one another and in which almost all of the interchange of momentum between adjacent layers is caused by the irregular motion of small fluid masses.

In order to discuss the boundary layer it is first necessary to define the terms velocity profile, surface friction, boundary-layer thickness, and separation point. A velocity profile is the curve that gives the distribution of the component of velocity parallel to the surface with distance normal to the surface (fig. 1). In boundary-layer theory the velocity component parallel to the surface is equal to the magnitude of the total velocity because the velocity component normal to the surface is negligible. The surface friction is the shearing stress between the fluid and the solid body.

The thickness of the boundary layer may be defined as the distance, in direction normal to the surface, at which the total pressure $\left(p + \frac{\rho U^2}{2}\right)$ differs by an arbitrary small amount from the total pressure of the undisturbed flow. At distances from the body greater than δ, the flow is assumed to be inviscid (fig. 2).

The separation point is the point on the surface of the body at which the surface friction is zero. Upstream of the point the direction of flow in the boundary layer next to the surface is downstream, and downstream of the point the direction of flow in the boundary layer next to the surface is upstream (fig. 3).

LAMINAR BOUNDARY LAYER

Methods of Calculation

By the use of the boundary-layer equations of motion together with the conditions that the solutions of the equations must satisfy at the inner and outer edges of the boundary layer the characteristics of the laminar boundary layer over a body may be determined completely when the velocity distribution over the body outside the boundary layer is known and the flow is such that the boundary-layer approximations are applicable. In order to avoid the purely mathematical difficulties associated with the exact method of solution, various approximate methods for the computation of the velocity profile, surface friction, boundary-layer thickness, and separation point have been developed.

Von Kármán momentum equation.- The von Kármán momentum equation (reference 3) results from the application of the momentum theorem to boundary-layer flow and makes it possible to compute the boundary-layer thickness over a body with an arbitrary pressure distribution whether the flow in the boundary layer is laminar or turbulent if the variation of the boundary-layer velocity profile and surface friction over the body is known.

The equation is obtainable by two methods: by applying the momentum principle to a box, the lower side of which extends for an infinitesimal distance along the solid surface, the upper side of which is the nominal thickness of the boundary layer, and the sides of which are planes perpendicular to the body surface (appendix), or by integrating the boundary-layer equation of motion with respect to the distance normal to the surface. The von Kármán momentum equation contains the same assumptions as the Prandtl boundary-layer equations (appendix) and may be written as (appendix)

$$\rho \frac{du^2\theta}{dx} = \delta^* \frac{dp}{dx} + \tau_o \tag{1}$$

The quantity θ, where

$$\rho U^2\theta = \rho U \int_0^\delta u \, dy - \rho \int_0^\delta u^2 \, dy$$

is called the momentum thickness and is a length of such magnitude that $\rho U^2\theta$ represents the difference between the rate of momentum flow that would exist if the mass flowing through a boundary-layer cross section were moving with the velocity at the boundary-layer edge and the actual rate of momentum flow through the boundary-layer cross section.

The quantity δ^*, where

$$\rho U\delta^* = \rho U\delta - \rho \int_0^\delta u \, dy$$

is called the displacement thickness and is a length of such magnitude that fluid flowing through it with the boundary-layer edge velocity U produces a rate of mass flow equal to the difference between the rate of mass flow which would exist if the fluid through a cross section of the boundary layer were flowing with the boundary-layer edge velocity and the actual rate of mass flow through the cross section of the boundary layer.

Equation (1) indicates that the effect of pressure gradient on the momentum defect of the fluid flowing through the boundary layer is directly proportional to the displacement thickness of the boundary layer.

For use in the Pohlhausen method and for the computation of boundary-layer thicknesses, the von Kármán equation is written as

$$\frac{d\theta}{dx} + \frac{\frac{\delta^*}{\theta} + 2}{U}\frac{dU}{dx} = \frac{\tau_o}{\rho U^2} \tag{2}$$

<u>Pohlhausen's method</u>.- The Pohlhausen method, an approximate method based on the von Kármán momentum equation has been widely used and is useful for obtaining qualitative information. The purpose of the Pohlausen method (reference 1, pp. 108-112) is to compute the characteristics of the laminar boundary layer in two-dimensional flow when the pressure distribution outside the boundary layer is a known function of the distance along the surface. The method is based on the assumption that all laminar boundary-layer velocity profiles are given by a fourth-degree polynomial. Pohlhausen chose a fourth-degree polynomial after trying first-, second-, and third-degree polynomials, because the fourth-degree polynomial gave better agreement between his method and the exact Blasius solution for the velocity profile and skin friction on a flat plate than polynomials of lower degree. The coefficients of the fourth-degree polynomial are chosen to make the equation for the velocity profile satisfy the boundary-layer equation of motion at the inner and outer edges of the boundary layer. The result is an equation for the velocity profile $\frac{u}{U} = f\left(\lambda, \frac{y}{\delta}\right)$ where $\lambda = \frac{\delta^2}{\nu}\frac{dU}{dx}$. (See appendix.) The velocity profiles are thus a single-parameter family of curves in which the parameter λ depends on the previous history of the boundary layer only through the thickness δ. The parameter λ is proportional to $-\frac{\delta \frac{dp}{dx}}{\tau_o}$ (appendix), the ratio of the unbalanced

horizontal pressure force acting on the small box used in deriving the von Kármán momentum equation to the shearing stress acting on the surface side of the box. The parameter λ can be shown to be independent of R_c (appendix). For the interior of the boundary layer the equation of motion is ignored, but the von Kármán momentum equation is satisfied. By satisfying the von Kármán momentum equation, the distribution over the body surface of the velocity profile, boundary-layer thickness, surface friction, and surface-pressure distribution is made consistent with the momentum theorem.

To obtain the boundary-layer characteristics over the surface, the differential equation resulting from the substitution of the equation for the velocity profile into the von Kármán equation is solved (reference 1, pp. 108-112). The method does not suffer from serious inaccuracies for $0 \leq \lambda < 12$. For flow over a flat plate, $\lambda = 0$, the skin friction differs from the exact value by only $3\frac{1}{2}$ percent. When, however, the pressure outside the boundary layer rises in the direction of flow, $\lambda < 0$, the method becomes inaccurate. An investigation of the reason for the inaccuracy of the Pohlhausen method (reference 4) leads to the conclusion that an inherent characteristic of the method is the late prediction of the separation point because the fourth-degree polynomial for the velocity distribution is not a good methematical substitute for the actual velocity profiles obtained in the exact solutions. Although the method is inaccurate in an adverse pressure gradient, it may often be used to obtain rapidly qualitative results concerning the effects on the velocity profile and surface friction of changes in the pressure distribution or boundary-layer thickness (appendix). The method is an example of the fact that a theory which ignores the equation of motion in the interior of the boundary layer and therefore neglects the acceleration terms in the equation of motion will make the boundary-layer profile dependent on only the local conditions. A complete theory would make the space rate of change of boundary-layer profile, rather than the profile itself, depend on the local conditions.

Falkner's method.- The purpose of Falkner's method (references 4 and 5) is to provide for the practical computation of the surface friction, momentum thickness, and displacement thickness of any two-dimensional laminar boundary layer. The method is based on existing tables of solutions of the boundary-layer equation for special types of pressure distribution (reference 4). In order to obtain the tables, Falkner expanded the quantity $\phi = \left(\frac{\tau_o}{\rho U^2}\right)\left(\frac{Ux}{\upsilon}\right)^{1/2}$ in a Taylor series from the stagnation point with x as the independent variable.

The coefficients in the Taylor series are the derivatives of ϕ with respect to x and were obtained from the derivatives of the complete boundary-layer equation at the stagnation point, with the help of known solutions of the special equation (reference 4) to which the complete equation reduces in the vicinity of the stagnation point. The series expansion for ϕ was then limited to a special type of pressure distribution given by $U = kx^{n}e^{xX'}$, where X' is a constant. The separation point $(\phi = 0)$ for the special type of pressure distribution was determined by using the Taylor series expansions for ϕ. After the separation points were computed, the solutions for different pressure distributions of the same family were tabulated and completed; these tables form the basis of the simplified method of calculation (reference 5).

The assumptions in Falkner's simplified method are: (1) The special form of the boundary-layer equation of motion represents the conditions of flow near the stagnation point with good accuracy. (2) The surface friction at a point is given accurately by replacing the actual pressure distribution by one of the particular family of pressure distributions; a new pressure distribution is chosen for each point. (3) The relation $H = \frac{\delta^*}{\theta}$ is a function only of ϕ, and this function can be determined from solutions of the boundary-layer equation of motion near the stagnation point.

The computation that must be made to solve a problem is simple and rapid. It consists of evaluating a simple integral (reference 5) from the given data and using the results of the integrations with the standard tables to obtain all the quantities that are of interest.

The method is shown to be suitable for the computation of the separation point by the good agreement between the computed separation point and the known separation point for two cases: One, an exact solution of the boundary-layer equations and the other, an experimental determination of the separation point (reference 5).

<u>Hartree's method</u>.- The purpose of Hartree's method is to provide an accurate method for the computation of all the characteristics of the laminar boundary layer. The basis of the method is the integration of the boundary-layer equation of motion by the replacement of the equation of motion, a partial differential equation, by an equation involving finite differences and ordinary derivatives. The only approximation used, other than those contained in the boundary-layer equation, is that a derivative may be replaced by a finite difference.

In order to compute the boundary-layer characteristics from the given pressure distribution, the partial differential equation of motion is replaced by an approximately equivalent ordinary differential equation by replacing the derivatives with respect to one of the variables by corresponding finite-difference ratios. The derivatives with respect to the other variable are integrated either mechanically or by some standard process for the numerical integration of ordinary differential equations. Derivatives parallel to the boundary are replaced by finite differences, and integration is carried out along successive normals to the boundary at finite intervals so that from the distribution of velocity across one section of the boundary layer the distribution of velocity across another section at an interval downstream is calculated. The limit to the accuracy obtainable with the method is the amount of work, which increases with the accuracy desired.

Critical Remarks

The Pohlhausen method is useful for obtaining general qualitative information for either favorable or adverse pressure gradients and for obtaining quantitative information for favorable pressure gradients. Falkner's method seems to provide sufficient accuracy for the solution of problems and to be rapid. Hartree's method is potentially more important than any of the others because if the computational work can be decreased the method can provide an accurate solution of the boundary-layer equations. At present the method is useful for providing solutions for testing approximate methods. The methods of references 6 to 12 seem to be inferior to Falkner's rapid method for general use because they either take much longer or are not so accurate.

When rapidity of computation is of prime importance, the position of the separation point of the laminar boundary layer may be estimated by replacing the velocity distribution over the body in the region of adverse gradient by a velocity distribution with a constant gradient. Approximate methods based on the use of a constant gradient may be obtained from the Pohlhausen method (reference 1, pp. 108-112), from the von Kármán-Millikan method (references 7 and 8), and from the work of Howarth (reference 6). A method based on Pohlhausen's work will usually predict separation too far downstream and the methods based on the von Kármán-Millikan method will usually predict separation too far upstream. The accuracy to be expected from a method based on Howarth's exact solution is unknown but should be good when the actual gradient is close to constant.

Exact Solutions of Boundary-Layer Equations

Blasius solution of boundary-layer equation for flow over a flat plate with zero pressure gradient.- The results obtained from the Blasius solution (reference 1, pp. 84-90) are

(1) The thickness of the boundary layer is proportional to $\sqrt{x\nu/U_o}$.

(2) The drag coefficient based on the drag of one side of a flat plate and the projected area of the plate is $C_D = \dfrac{1.328}{\sqrt{R_x}}$.

(3) The velocity ratio u/U_o in the boundary layer is a function of the single variable $\eta = \dfrac{y}{\sqrt{x\nu/U_o}}$.

Howarth.- Howarth (reference 6) solved the boundary-layer equations of motion for the case of flow over a flat plate with the velocity outside the boundary layer decreasing linearly in the direction of flow and with zero thickness of the boundary layer at the plate leading edge (fig. 4). The results obtained were

(1) The thickness of the boundary layer depends only on $\sqrt{x\nu/U_o}$ and on $\left|\dfrac{dU}{dx}\right| \dfrac{x}{U_o}$.

(2) The velocity ratio u/U_o in the boundary layer depends only on $\dfrac{y}{\sqrt{x\nu/U_o}}$ and on $\left|\dfrac{dU}{dx}\right| \dfrac{x}{U_o}$.

(3) The local surface-friction coefficient $\tau_o/2q$ decreased from an extremely large value at the leading edge to zero at the separation point.

(4) The amount of velocity recovery $\Delta U/U_o$ before separation (fig. 4) is a constant and is independent of the Reynolds number and of the rapidity with which the velocity is recovered. The rapidity of velocity recovery does not appear because the initial boundary-layer thickness is zero.

Von Kármán.- Von Kármán reduced the complete equations of motion for a rotating disk in laminar flow to a system of ordinary differential equations and obtained approximate solutions (reference 3). The results were

(1) The boundary-layer thickness is constant over the rotating disk and is $\frac{\delta}{a} = 2.58\sqrt{\frac{\upsilon}{\omega a^2}}$.

(2) The inflow velocity normal to and far from the disk is $C = 0.708\sqrt{\upsilon\omega}$.

(3) The turning moment required to rotate the disk for both sides is $M = \frac{3.68}{\sqrt{\frac{\omega a^2}{\upsilon}}} a^3\rho \frac{(\omega a)^2}{2}$.

Cochran (reference 2) solved the system of ordinary differential equations exactly by a numerical process. As in any exact solution of the equations of motion or of the equations of the boundary layer, no definite boundary-layer thickness was obtained. The constant in result (2) for C was found to be 0.866 instead of 0.708 and the constant in result (3) for M was found to be 3.87 instead of 3.68.

Conclusions from Prandtl boundary-layer equation.- Useful information can be obtained from the Prandtl boundary-layer equation as given by Falkner in reference 4 without obtaining solutions of the equation. The conclusions are

(1) For a fixed velocity distribution along the body and a fixed point on the body, the nondimensional thickness δ/c of the boundary layer is inversely proportional to $\sqrt{\frac{U_o c}{\upsilon}}$ (fig. 5).

(2) For a fixed velocity distribution along the body and a fixed point on the body, the local surface-friction coefficient $\frac{\tau_o}{2q}$ is inversely proportional to $\sqrt{\frac{U_o c}{\upsilon}} = \sqrt{R_c}$. The total drag coefficient of the part of a body covered by a laminar boundary layer therefore varies as $1/\sqrt{R_c}$.

(3) For a given velocity distribution over the body, the separation point is independent of the Reynolds number R_c. This fact is valuable in experimental work and in cases where computations of boundary-layer characteristics are made for more than one Reynolds number.

(4) For a fixed velocity distribution along the body and a fixed point on the body, the curve of u/U against y/δ is invariable and is independent of the Reynolds number. The curve of u/U against $\frac{y}{c}\sqrt{R_c}$ is invariable; this provides a good method for testing whether a velocity distribution is laminar.

TURBULENT BOUNDARY LAYERS

In contrast to the smooth flow associated with laminar boundary layers, a mixing flow is generally associated with turbulent boundary layers. In a turbulent boundary layer the momentum interchange between adjacent fluid layers is caused mainly by the irregular motion of small fluid masses. The information concerning turbulent flow is largely empirical; whereas the information for laminar motion is obtained wholly from the equations of motion.

Skin friction.- Empirical skin-friction formulas for flow over smooth flat plates with zero pressure gradient and for flow in pipes where a small favorable pressure gradient exists are available (reference 1, pp. 135-154, and references 13 to 15). Empirical skin-friction formulas are also given by Goldstein in a British paper of limited distribution. These references show that

(1) For equal Reynolds numbers the turbulent skin-friction coefficient is greater than the laminar skin-friction coefficient.

(2) The turbulent skin-friction coefficient decreases less rapidly than the laminar skin-friction coefficient as the Reynolds number increases.

The turbulent skin-friction coefficient can be calculated from the same relation as for laminar flow $\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_0$, if the velocity profile is known inside the laminar sublayer, that is, the region at the wall in which the velocity fluctuations disappear and the flow is laminar (reference 18). An estimate of the thickness of the laminar sublayer, based on data from flow through pipes, is given by the relation $\frac{\delta_* U}{\nu} \approx \frac{11.6}{\sqrt{\tau_o/2q}}$ (reference 15).

Shearing stress.- No exact relation is known between the derivative of the average velocity at a point inside a turbulent boundary layer and the shearing stress at the point. The absence of this information forces recourse to experiment to obtain information concerning turbulent flow. If a relation between the local shearing stress and the derivative of the average local velocity is assumed a method for computing the characteristics of the turbulent boundary layer may be devised. The method, however, will depend on the assumption concerning the shearing stress.

From the fundamental relation for the shearing stress in turbulent flow $\tau = -\rho \overline{u'v'}$ (reference 1, pp. 119-134, and reference 15), Prandtl derived the mixing-length theory, an approximate theory that relates the local shearing stress to the local density, the local derivative of the average velocity, and the local value of a so-called mixing length l (reference 15). The equation is $\tau = \rho l^2\left(\frac{\partial u}{\partial y}\right)\left|\frac{\partial u}{\partial y}\right|$. The mixing-length theory is based on the fact that if two adjacent layers of fluid have different velocities, the interchange between the two layers of small masses of fluid that have the velocities of the layers from which they come will tend to equalize the velocities of the fluid layers. The tendency toward equalization of velocities may be considered as being caused by an apparent shearing stress between the two layers. Von Kármán (reference 15) obtained an approximate expression for the mixing length as a function of the local velocity derivatives and a universal constant by assuming that the process of turbulence at any two points is similar and differs only in the length and time scales. The relation can be used to compute a velocity distribution in pipes that agrees very well with the experimental velocity distribution.

Boundary-layer velocity profile.- The velocity profile for the turbulent boundary layer differs markedly in appearance from the velocity profile of the laminar boundary layer (fig. 6). At large boundary-layer Reynolds numbers the turbulent velocity profile shows an extremely rapid rise in velocity in a very short distance. This large slope at the wall causes the turbulent skin-friction coefficient to be higher than the laminar skin-friction coefficient.

When the pressure gradient along the surface is zero or very slightly favorable, such as on plates or in pipes, the velocity profile can be approximated by a logarithmic curve. The equation is (reference 15)

$$\frac{u}{v_*} = A \log \frac{yv_*}{\upsilon} + B$$

and follows from the assumptions that the shearing stress is constant across the pipe or boundary layer and equal to the wall shearing stress and that the mixing length is given by the equation

$$l = Ky$$

For a wider range of pressure gradients including those which are adverse (that is, those in which the static pressure rises in the direction of flow), the velocity profile can be approximated by a power curve

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/m}$$

Both of these equations for the velocity distribution become inaccurate at distances from the wall comparable to the thickness of the laminar sublayer. At the wall both equations incorrectly give infinite values of du/dy.

If the flow continues in an adverse pressure gradient, the velocity profile undergoes a change from one having high velocities near the surface to one having low velocities near the surface (fig. 7 and reference 17).

For some purposes it seems permissible to assume that the turbulent boundary-layer profiles form a single-parameter family of curves and to use the ratio $H = \frac{\delta^*}{\theta}$, or the velocity ratio u/U at some fraction of the boundary-layer thickness from the surface, as the parameter. The turbulent velocity profiles usually found in undisturbed flow are simple curves and may be specified by any one of a number of suitably chosen parameters. The suggestion that turbulent boundary-layer profiles form a single-parameter family of curves first appeared in a paper by Gruschwitz (reference 18) in which the factor $\eta_1 = 1 - \left(\frac{u_\theta}{U}\right)^2$ was used as a parameter.

Separation point.- In an adverse pressure gradient the turbulent boundary layer will eventually separate. For similar conditions, however, the percentage of the initial dynamic pressure which can be converted into static pressure in flow over surfaces covered by turbulent boundary layers is greater than in flow over surfaces covered by laminar boundary layers. Because the mixing between the inner and outer layers of fluid is much greater in a turbulent boundary layer than in a laminar boundary layer, more downstream momentum is transferred from the outer layers of fluid to the inner layers of fluid. The downstream momentum of the fluid near the wall in turbulent flow is therefore maintained in cases in which the downstream momentum of laminar flow would be exhausted by the surface shear and separation would occur.

Only empirical methods are available for estimating the separation point of turbulent boundary layers. Of all the methods (reference 1, pp. 155-162; references 17 to 22; and Garner's method which is given in a British paper of limited distribution) the two which seem to be most useful for predicting the behavior of the turbulent boundary layer are those of Garner and reference 17. The Gruschwitz method (reference 18) introduced the idea of a single-parameter family of curves for the velocity profiles of the turbulent boundary layer and made the rate of change along the surface of the shape parameter rather than the shape parameter itself depend on the local conditions but did not make possible the computation of the separation point with sufficient accuracy for engineering use (reference 23). An attempted improvement of the Gruschwitz method by Kehl (reference 20) has not been tested for its ability to predict the separation point.

The method of reference 17 seems to be the most reliable method available at present for the estimation of the separation point of the turbulent boundary layer. The method uses two equations to determine the behavior of a turbulent boundary layer. The first equation is the von Kármán momentum equation (equation (2)). The second equation is an empirical equation that gives the rate of change of boundary-layer shape parameter along the surface as a function of the local conditions. The equation for the rate of change of boundary-layer shape parameter was developed in the following manner:

It was first verified that for the experimental data available, the velocity profiles of the turbulent boundary layer formed a single-parameter family of curves with $H = f(\eta_1)$ as the parameter. The assumption was then made that the rate of change of boundary-layer shape parameter is a function of the ratio of the local pressure gradient $\frac{\theta}{q}\frac{dq}{dx}$ to the local skin-friction coefficient $\tau_o/2q$ and also to the local value of H. The ratio $\frac{\frac{\theta}{q}\frac{dq}{dx}}{\tau_o/2q}$ has the same physical significance as the parameter λ in the Pohlhausen method. The quantities $\frac{\theta}{q}\frac{dq}{dx}$ and H were determined from the available experimental data; the term $\tau_o/2q$ was calculated from the skin-friction formula of reference 24. From analysis of the experimental data, the variation of $\theta\frac{dH}{dx}$ with $\frac{\theta}{q}\frac{dq}{dx}\frac{2q}{\tau_o}$ and H can be represented by the equation

$$\theta\frac{dH}{dx} = e^{4.680(H-2.975)}\left[-\frac{\theta}{q}\frac{dq}{dx}\frac{2q}{\tau_o} - 2.035(H - 1.286)\right] \qquad (3)$$

Equation (3) is solved simultaneously with the von Kármán momentum equation by numerical methods. For a fixed pressure distribution and transition point the method indicates a slight forward movement of the separation point with increase in Reynolds number.

The method of Garner differs from that of reference 17 only by the use of a different empirical skin-friction relation and by the use of different constants in equation (3). The different constants were obtained by analyzing the experimental data in reference 17 and adding a small amount of data from experiments by Buri and Nikuradse.

None of the methods give any information on the variation of surface friction with boundary-layer profile shape; all use empirical skin-friction formulas derived from experiments with flat plates.

Momentum thickness.- An approximate method for the computation of boundary-layer momentum thicknesses that is useful for the estimation of full boundary-layer thicknesses and profile-drag coefficients when flow separation is not involved is given in reference 25. The method is based on the fact that if H is fixed at an average value and that if a skin-friction equation of the form $\frac{\tau_o}{2q} = \frac{D}{R_\theta{}^{g}}$ is used, the von Kármán momentum equation can be integrated. The result is a formula for the computation of θ in flows with pressure gradients.

The same type of formula is applicable to laminar boundary layers when the average velocity gradient is small. The formula for laminar boundary layers has also been given in reference 26.

Roughness.- Empirical skin-friction formulas for flow over rough flat plates and for flow in rough pipes are available for certain types of roughness (reference 1, pp. 145-154 and references 2, 15, and 27). A characteristic of the flow over rough surfaces with zero or small pressure gradient is that beyond a certain Reynolds number, which depends on the roughness, the skin-friction coefficient becomes independent of the Reynolds number (reference 28). The skin-friction coefficient of rough rotating cylinders (reference 30) was found to become constant at sufficiently large Reynolds numbers for a saturation density of roughness particles. For other than saturation densities, the drag coefficient was concluded to decrease with Reynolds number.

The addition of roughness to a surface covered by a turbulent boundary layer increases the drag coefficient when the roughness height becomes comparable with the height of the laminar sublayer. The addition of roughness to a smooth plate will have no effect on the surface friction if $R_{\epsilon}\sqrt{\frac{\tau_o}{2q}} \lessapprox 3$, where $\tau_o/2q$ is the surface-friction coefficient for the smooth plate (reference 30).

TRANSITION

A body in a stream usually has a laminar boundary layer for some distance from the stagnation point and behind the laminar boundary layer, a turbulent-boundary layer that extends to the trailing edge. The process of change of the laminar boundary layer to the turbulent boundary layer is known as transition. The appearance of the turbulent type of flow can be detected by the appearance of random fluctuations in the velocity components, by the change in velocity profile from one having a gradual increase in velocity with distance from the walls to one having a much more rapid rise in velocity (fig. 6), and by the increase in skin-friction coefficient; a greater increase occurs in skin-friction coefficient at large Reynolds numbers than at small Reynolds numbers.

The flow conditions that are favorable for the delay of transition are: a small Reynolds number, freedom from disturbances, and static pressure decreasing in the direction of flow (references 31 and 32). An investigation of the effect of curvature (reference 33) results in the following conclusions: The mechanism of transition is different on concave and convex walls, the transition point is not affected by convex curvature, concave curvature has a strong destabilizing effect, the influence of pressure gradient on transition on a concave wall is negligible, the effect of pressure gradient on transition on convex walls is strongest near zero pressure gradient, and stream turbulence has about the same effect on transition for flow over concave and convex walls.

When the static pressure increases in the direction of flow, the maximum possible length of laminar flow is the distance between the minimum pressure point and the laminar separation point. Whether transition or laminar separation occurs first depends on the Reynolds number, the disturbances to the flow, and the strength of the adverse pressure gradient. As the adverse pressure gradient becomes smaller, the likelihood of transition occurring before separation becomes greater; the flat plate represents the extreme case in which separation never occurs.

The instability of the laminar boundary layer on a flat plate has been investigated theoretically (reference 34) and the essentials of the theory have been verified experimentally (reference 35). A process for the computation of the instability point of the laminar boundary layer in the presence of pressure gradients, with examples of the results for airfoils, is given in reference 36. Because it takes some distance for the laminar flow to become turbulent after passing the instability point, the transition point is downstream of the instability point. A comparison between the experimental transition points and the theoretical instability points for an NACA airfoil is given in references 36 and 37. The comparison shows that, as is expected from the theory, a decreasing static pressure in the direction of flow causes the distance between the instability and transition points to increase and also causes the instability point to move farther downstream at a fixed Reynolds number.

REGIONS OF SEPARATED FLOW ON AIRFOILS

At sufficiently small airfoil Reynolds numbers a region of separated flow is often found that has for its forward boundary the laminar separation point and for its rearward boundary a turbulent boundary layer (references 38 to 40). The usual places of occurrence of the region of separated flow are near the leading edge of airfoils at high angles of attack and behind the minimum pressure point on airfoils which have extensive laminar boundary layers.

Only qualitative information is available concerning the extent of the region of separated flow. The flow can reattach itself to the surface as a turbulent boundary layer at some small distance behind the point at which the laminar boundary layer leaves the surface. The extent of the region of separated flow lying between the laminar and turbulent boundary layers has been observed to decrease with an increase in Reynolds number (reference 40). No case is known for which the separated turbulent boundary layer has rejoined the surface in free flow.

The formation of turbulent velocity fluctuations in the fluid layers that are moving downstream and that are just above the region of separated flow is believed to be important in the formation of a turbulent boundary layer behind the region of separated flow.

PROFILE-DRAG COMPUTATION

The method of profile-drag computation is based on the momentum theorem which may be stated as follows: In a steady flow without body forces the net external force acting in a particular direction on the surface bounding a fixed region of fluid is equal in magnitude to the difference between the time rate of outflow and time rate of inflow of momentum in the direction under consideration and has the sense in which the momentum decreases.

The profile drag can therefore be determined if the difference between the time rate of outflow and time rate inflow of momentum in the direction of flight can be determined for a region of fluid bounded by the body and a surface which has a shape and distance from the body so chosen that the pressures produce no resultant force on this surface in the line of flight direction. Squire and Young (reference 24) determine the profile drag by computing the momentum thickness at the airfoil trailing edge from the von Kármán momentum equation; then by the use of the von Kármán momentum equation together with an assumption concerning the velocity profile across the wake they compute the momentum thickness very far behind the body. The profile-drag coefficient is known once the momentum thickness, and therefore the momentum defect, is known very far behind the body.

The Squire and Young method is accurate to within a few percent if no regions of separated flow are present and if the transition point is known. It must be emphasized that the profile-drag coefficient can be computed only when the transition point is known and that the profile-drag coefficient is sensitive to the position of the transition point.

PIPE FLOW

The subject of flow in pipes has been studied exhaustively and the knowledge of the subject may be found in many works; for example, references 1, 2, and 15. The steady laminar flow in a pipe can be computed directly from the equations of motion. Included are the pressure-drop formulas and the velocity profile (reference 1, pp. 36-39).

The knowledge of turbulent flow in pipes, like that of turbulent flow over bodies, is based on experiment. The pressure-drop formulas have been determined from flow experiments with smooth-wall and rough-wall pipes (reference 1, pp. 135-145, and references 2 and 15). Universal velocity-distribution formulas have been determined for flow in smooth and rough pipes.

The mixing-length theories of Prandtl and von Kármán make possible the computation of the velocity distribution across the pipe when the surface friction is known.

DIFFUSERS

A diffuser is a duct having an internal area that increases with distance downstream. Because of the increasing area, the flow velocity in the diffuser decreases with distance downstream and therefore the static pressure in the flow increases.

A purely theoretical treatment of laminar flow in a two-dimensional diffuser is given in reference 2, which states that the results are of theoretical interest only.

The literature for diffusers with turbulent flow is extensive; but because no theory of turbulent flow has been established for diffusers, the work of the various experimenters (for example see reference 41) has not resulted in the ability to predict the behavior of a given diffuser with a good degree of certainty.

FREE-MIXING PROCESSES

A free-mixing process is one in which solid boundaries play no part. Some examples of cases in which free-mixing processes occur are jets, wakes, and regions in which parallel streams of different velocities meet. The cases of turbulent-mixing processes, treated in references 42 to 45, are based upon an assumption relating the local shearing stress to the characteristics of the local velocity profile. The equations and assumptions of these references are

(1) The equation of motion with zero static-pressure gradient (far from bodies)

(2) The equation of continuity

(3) The Prandtl equation for the shearing stress

$$\tau = \rho l^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial u}{\partial y} \tag{4}$$

(4) The mixing length l proportional to the width of the mixing region

The cases treated in reference 46 use for the shearing stress the equation

$$\tau = \rho l \left(U_{max} - U_{min}\right) \frac{\partial u}{\partial y} \tag{5}$$

instead of $\tau = \rho l^2 \left(\frac{\partial u}{\partial y}\right) \left|\frac{\partial u}{\partial y}\right|$. The sharpness of the calculated velocity profiles at extreme values of u is eliminated by using equation (5) instead of equation (4) but no better understanding of the flow process results.

Turbulent jets and wakes and turbulent flows in the presence of boundaries have also been analyzed by considering the velocity and pressure fluctuations in the flow. An introduction to this method of investigation is given in reference 47.

EFFECT OF BOUNDARY LAYER ON POTENTIAL-FLOW CHARACTERISTICS OF AIRFOILS

Airfoil characteristics computed by potential-flow theory are known to differ from the experimentally determined characteristics. The deviation from the potential-flow characteristics appears not only in the existence of a profile drag but also in changes in the lift and pitching-moment characteristics. The changes are caused by the presence of the boundary layer and usually increase with increasing boundary-layer thickness.

The basis of the method of computation of reference 48 is Taylor's theorem which states that equal positive and negative amounts of vorticity are shed from the airfoil trailing edge per unit time when the lift is steady. The method of computation of reference 48 in outline is approximately as follows: The potential-flow velocity distribution over the airfoil is found; a suitable fairing is made at the trailing edge if necessary to avoid a stagnation point. The boundary-layer thicknesses at the trailing edge of the upper and lower surfaces are determined after choosing the transition point and the Reynolds number. The velocities at the edge of the upper-surface and lower-surface boundary layers at the trailing edge are then computed; when these velocities are known the pressure rise through the boundary layer is estimated for both upper and lower surfaces. If the pressure at the trailing edge is not the same for both upper and lower surfaces, Taylor's theorem is violated; the circulation is therefore adjusted and the procedure repeated until the pressures are equal. At present the computations are too lengthy and, without empirical correction factors, are too inaccurate for routine use.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., May 21, 1947

APPENDIX

MATHEMATICAL DERIVATIONS

Prandtl Boundary-Layer Equations

The equations of motion in Cartesian coordinates for incompressible flow are as follows:

The equation of motion for x-direction is

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).$$

The equation of motion for y-direction is

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

For the boundary layer, the equations become the Prandtl boundary-layer equations; thus

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \tag{A1}$$

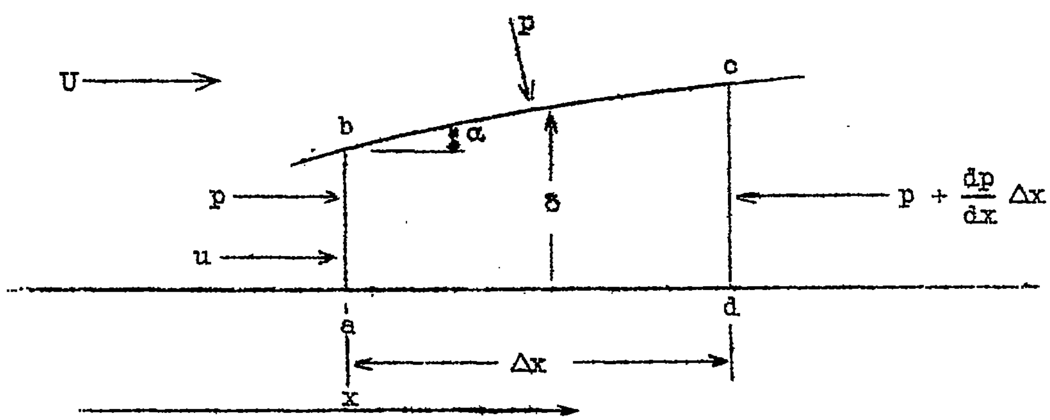
$$0=\frac{\partial p}{\partial y} \tag{A2}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Equations (A1) and (A2) are valid when the pressure is constant across the thickness of the boundary layer, when the ratio of the boundary-layer thickness to the curvature of the surface is negligible, and when all viscous terms involving either v or derivatives with respect to x are negligible. For flows about airfoils at normal angles of attack and about plates at zero angle of attack these conditions are accurate over most of the length of the surface. Regions where the conditions may not be accurate are in the vicinity of stagnation points and in the vicinity of the separation point.

Derivation of the von Kármán Momentum Equation



Continuity

The mass leaving box abcd equals the mass entering box abcd. The mass entering per unit time through ab is equal to

$$\int_0^\delta \rho u \, dy$$

The mass leaving per unit time through cd is equal to

$$\int_0^\delta \rho u \, dy + \Delta x \frac{d}{dx} \int_0^\delta \rho u \, dy$$

Therefore, the mass entering per unit time through bc is equal to

$$\left(\int_0^\delta \rho u \, dy + \Delta x \frac{d}{dx}\int_0^\delta \rho u \, dy\right) - \left(\int_0^\delta \rho u \, dy\right) = \Delta x \frac{d}{dx}\int_0^\delta \rho u \, dy$$

Application of Momentum Theorem

The net change in momentum per unit time in the x-direction equals the net force acting to right on box in the x-direction. The x-momentum entering per unit time through ab is given by

$$\int_0^\delta \rho u^2 \, dy$$

The x-momentum entering per unit time through bc is given by

$$U \, \Delta x \frac{d}{dx}\int_0^\delta \rho u \, dy$$

The x-momentum leaving through cd is given by

$$\int_0^\delta \rho u^2 \, dy + \Delta x \frac{d}{dx}\int_0^\delta \rho u^2 \, dy$$

Net Excess of x-Momentum Leaving to Right over That Entering from Left

The net excess of x-momentum leaving to right over that entering from left is given by

$$\int_0^\delta \rho u^2 \, dy + \Delta x \frac{d}{dx}\int_0^\delta \rho u^2 \, dy - \int_0^\delta \rho u^2 \, dy - U \, \Delta x \frac{d}{dx}\int_0^\delta \rho u \, dy$$

Net Forces Acting to Right on Box

The pressure acting to the right on the box is given by

$$p\delta + p\,\frac{\Delta x}{\cos\alpha}\,\sin\alpha$$

The pressure acting to the left on the box is given by

$$p\delta + \frac{dp\delta}{dx}\,\Delta x$$

The skin friction to the left on the box is given by

$$\tau_o\,\Delta x$$

Therefore, the net force acting to the right on the box is given by

$$p\delta + p\,\Delta x\tan\alpha - p\delta - \frac{dp\delta}{dx}\,\Delta x - \tau_o\,\Delta x$$

Then, since the net change in momentum in the x-direction equals net force acting to right on box,

$$\Delta x\left(\frac{d}{dx}\int_0^\delta \rho u^2\,dy - U\,\frac{d}{dx}\int_0^\delta \rho u\,dy\right)$$

$$= p\,\Delta x\tan\alpha - \delta\,\frac{dp}{dx}\,\Delta x - p\,\frac{d\delta}{dx}\,\Delta x - \tau_o\,\Delta x$$

where

$$\tan\alpha = \frac{d\delta}{dx}$$

Then

$$\Delta x\left(\frac{d}{dx}\int_0^\delta \rho u^2\,dy - U\frac{d}{dx}\int_0^\delta \rho u\,dy\right) = -\Delta x\left(\delta\frac{dp}{dx} + \tau_o\right)$$

or

$$\frac{d}{dx}\int_0^\delta \rho u^2\,dy - U\frac{d}{dx}\int_0^\delta \rho u\,dy = -\delta\frac{dp}{dx} - \tau_o \qquad \text{(A3)}$$

After the definitions

$$U^2\theta = \int_0^\delta u(U - u)\,dy$$

and

$$U\delta^* = \int_0^\delta (U - u)\,dy$$

have been used, equation (A3) can be written as

$$\rho\frac{dU^2\theta}{dx} = \delta^*\frac{dp}{dx} + \tau_o \qquad \text{(A4)}$$

If the equation of motion for inviscid flow that is true outside the boundary layer, $\frac{dp}{dx} = -\rho U\frac{dU}{dx}$, is used and the term $\frac{dU^2\theta}{dx}$ is split into two terms, equation (A4) can be written as

$$\frac{d\theta}{dx} + \theta\left(\frac{H + 2}{U}\frac{dU}{dx}\right) = \frac{\tau_o}{\rho U^2}$$

The von Kármán equation is often written in this form. By using the Bernoulli equation to relate the static and dynamic pressure outside the boundary layer, $\frac{dp}{dx} = -\frac{dq}{dx}$, equation (A4) may also be written as

$$\frac{d\theta}{dx} + \frac{H + 2}{2} \frac{\theta}{q} \frac{dq}{dx} = \frac{\tau_o}{2q}$$

Derivation of the Expression for the Velocity Profile

in the Pohlhausen Method

Let the boundary-layer velocity profile be given by a fourth-degree polynomial

$$u = ay + by^2 + cy^3 + dy^4$$

The values of the coefficients are determined from the form taken by the equation of motion at the wall and from the assumption that all viscous effects are confined to a thin boundary layer.

At the wall, $u = v = 0$, and so the equation of motion

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2}$$

becomes

$$\left(\frac{\partial^2 u}{\partial y^2} \right)_{y=0} = \frac{1}{\mu} \frac{dp}{dx}$$

Because all viscous effects are assumed to be absent outside the boundary layer, $\tau = \frac{\partial \tau}{\partial y} = 0$ outside the boundary layer. Therefore, at $y = \delta$ with $\tau = \mu \frac{\partial u}{\partial y}$, it follows that $\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0$. From the definition of δ it also follows that $u = U$.

By using these four conditions, four simultaneous linear equations are obtained from which a, b, c, and d are evaluated. The result is

$$a = \frac{U(12 + \lambda)}{6\delta}$$

$$b = -\frac{U\lambda}{2\delta^2}$$

$$c = -\frac{U(4 - \lambda)}{2\delta^3}$$

$$d = \frac{U(6 - \lambda)}{6\delta^4}$$

where λ, the velocity profile shape parameter, equals $\frac{\delta^2}{\upsilon}\frac{dU}{dx}$. The expression for the velocity profile therefore becomes

$$\frac{u}{U} = \frac{12 + \lambda}{6}\left(\frac{y}{\delta}\right) - \frac{\lambda}{2}\left(\frac{y}{\delta}\right)^2 - \frac{4 - \lambda}{2}\left(\frac{y}{\delta}\right)^3 + \frac{6 - \lambda}{6}\left(\frac{y}{\delta}\right)^4$$

and the surface shearing stress $\tau_o = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is

$$\tau_o = \mu \frac{U}{\delta} \frac{12 + \lambda}{6}$$

or

$$\frac{\tau_o}{2q} = \frac{1}{R_\delta} \frac{12 + \lambda}{6}$$

In order to show that $\lambda \propto - \frac{\delta \frac{dp}{dx}}{\tau_o}$

use the definition of λ,

$$\lambda = \frac{\delta^2}{\upsilon} \frac{dU}{dx}$$

and

$$\frac{dp}{dx} = - \rho U \frac{dU}{dx}$$

It follows that

$$\lambda = - \frac{\delta^2}{\upsilon} \frac{1}{\rho U} \frac{dp}{dx} = - \frac{\delta \rho U}{\mu} \frac{\delta}{\rho U^2} \frac{dp}{dx} = - R_\delta \frac{\delta}{2q} \frac{dp}{dx}$$

Then, from

$$\frac{\tau_o}{2q} = \frac{1}{R_\delta} \frac{12 + \lambda}{6}$$

$$R_\delta \propto \frac{2q}{\tau_o}$$

and so

$$\lambda \propto -\frac{2q}{\tau_o}\frac{\delta}{2q}\frac{dp}{dx}$$

or

$$\lambda \propto -\frac{\delta\frac{dp}{dx}}{\tau_o}$$

To show that λ is independent of R_c

$$\lambda = \frac{\delta^2}{\upsilon}\frac{dU}{dx} = \left(\frac{\delta}{c}\right)^2 \frac{d\frac{U}{U_o}}{dx/c}\frac{U_oC}{\upsilon}$$

$$= \frac{d\frac{U}{U_o}}{dx/c}\left(\frac{\delta}{c}\right)^2 R_c$$

but (see p. 13)

$$\frac{\delta}{c} \propto \frac{1}{\sqrt{R_c}}$$

therefore,

$$\lambda \propto \frac{d\frac{U}{U_o}}{dx/c}$$

Consequently when the pressure distribution does not change with R_c, λ is independent of R_c.

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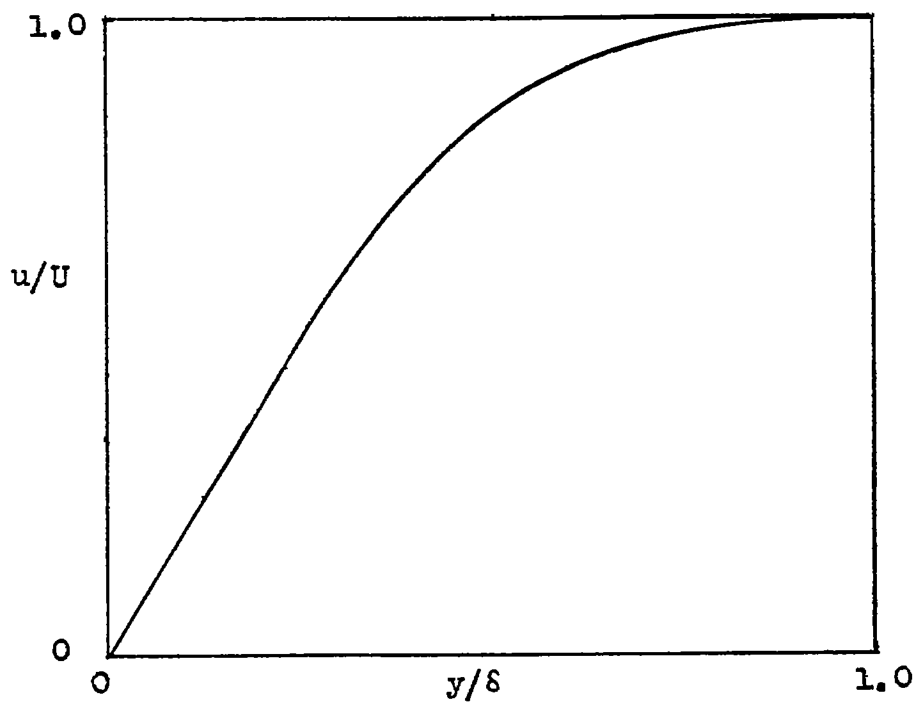

Figure 1.- Velocity profile.

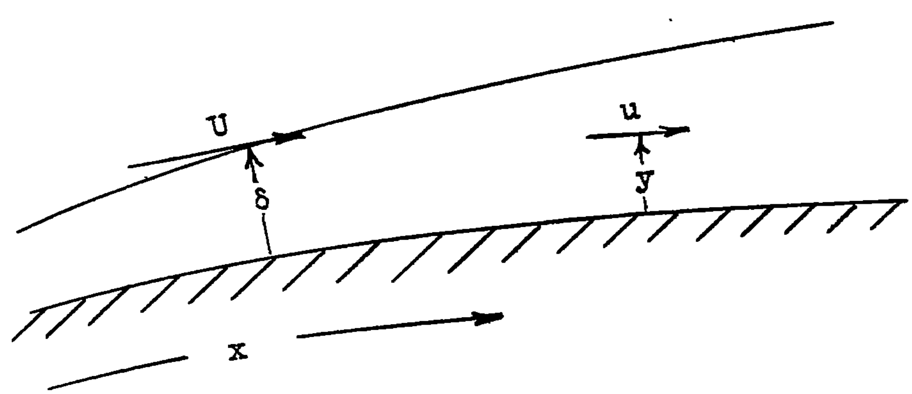

Figure 2.- Boundary layer.

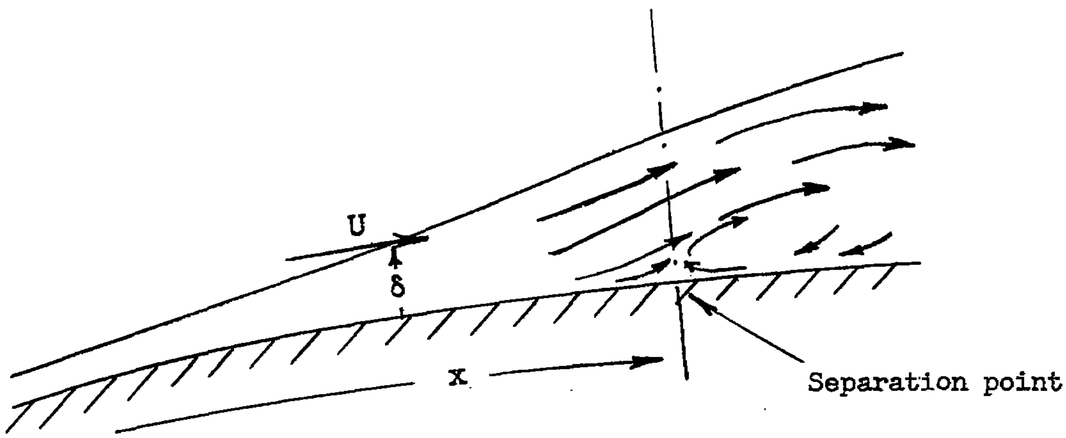

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Figure 3.- Separation point.

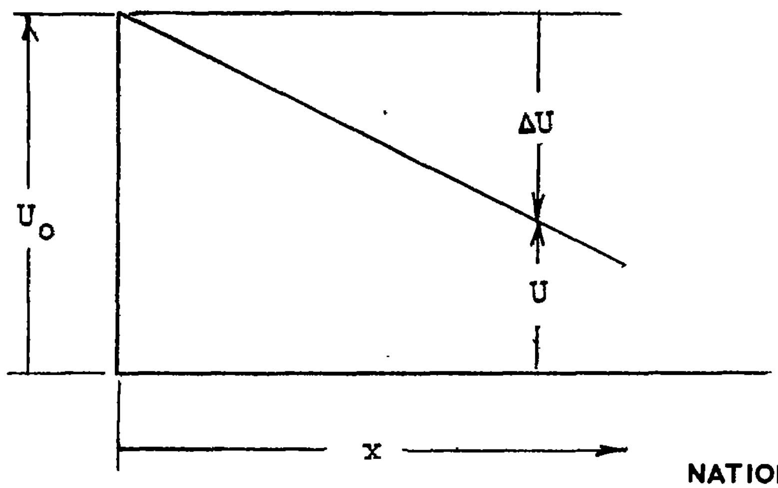

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Figure 4.- Velocity distribution. (See reference 6.)

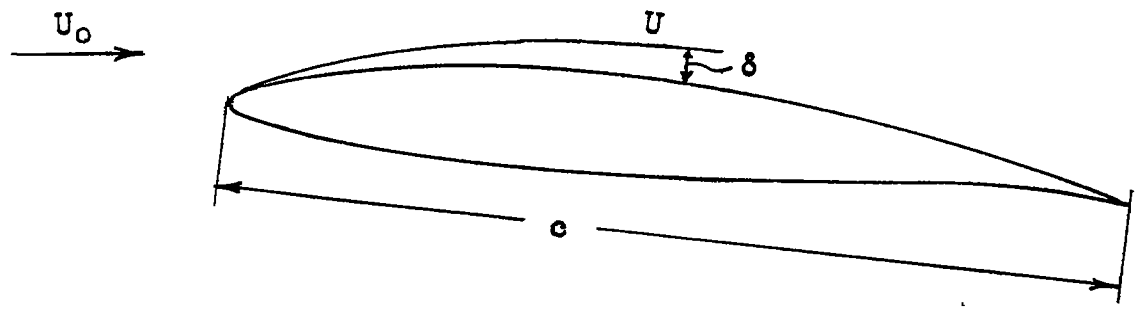

Figure 5.- Boundary-layer thickness on upper surface of airfoil.

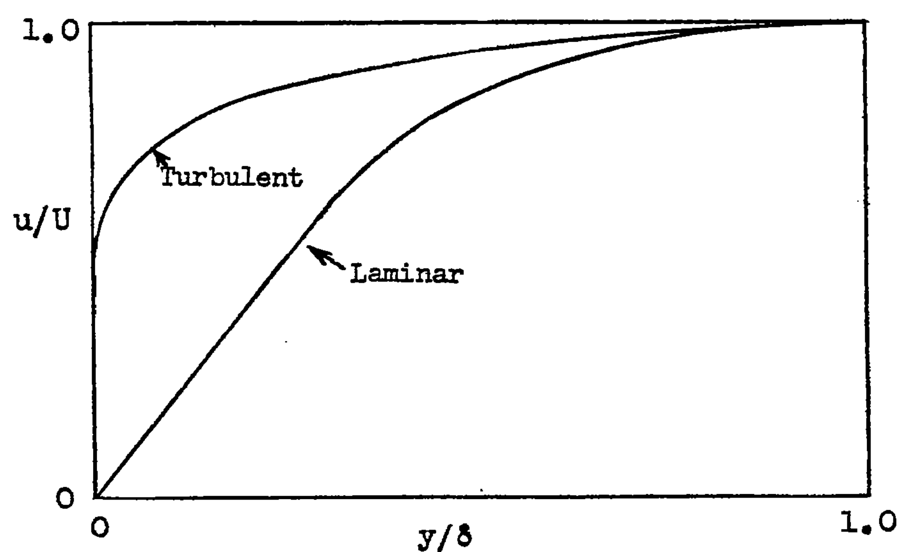

Figure 6.- Flat-plate turbulent and laminar velocity profiles.

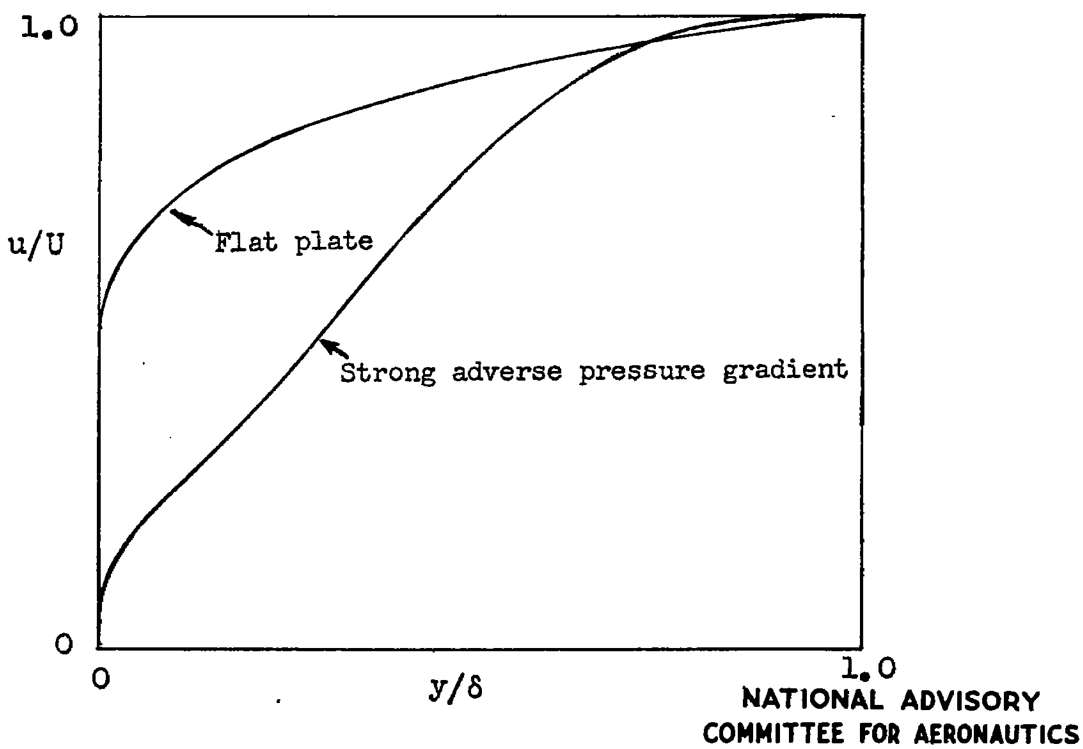

Figure 7.- Effect of adverse pressure gradient on velocity profile of turbulent boundary layer.

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